

On the connective eccentricity index of two types of trees*

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Abstract: The connective eccentricity index $\xi^{ce} = \sum_{u \in V} \frac{d(u)}{\varepsilon(u)}$, where $\varepsilon(u)$ and $d(u)$ denote the eccentricity and the degree of the vertex u , respectively. In this paper, we first determine the extremal trees which minimize and maximize the connective eccentricity index among all trees with a given degree sequence, and then determine the extremal trees which minimize and maximize the connective eccentricity index among all trees with a given number of branching vertices.

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1 Introduction

Let G be a simple and connected graph with $n = |V(G)|$ vertices. For a vertex $v \in V(G)$, $d_G(v)$ denotes the degree of v (or just $d(v)$ briefly). For vertices $v, u \in V(G)$, the distance $d(v, u)$ is defined as the length of a shortest path between v and u in G . The eccentricity $\varepsilon_G(v)$ (or just $\varepsilon(v)$ briefly) of a vertex v is the maximum distance from v to any other vertex of G . The diameter $D(G)$ of a graph is the maximum eccentricity of any vertex in the graph. A vertex of degree one is called a pendant vertex. A path $P = v_0v_1 \cdots v_t$ of G is a pendant path if v_0 is a pendant vertex, the degree of any internal vertex v_i ($1 \leq i < t$) is two and the degree of v_t is at least three. Let S_n and P_n denote the star and the path with n vertices, respectively. For other terminologies and notations not defined here we refer the readers to [1].

In 2000, Gupta, Singh and Madan [2] introduced a novel, adjacency-cum-path length based, topological descriptor termed the connective eccentricity index. In order to explore the potential of the connective eccentricity index in predicting biological activity, authors used nonpeptide N-benzylimidazole derivatives to investigate

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the predictability of the connective eccentricity index with respect to antihypertensive activity. They showed that results obtained using the connective eccentricity index were better than the corresponding values obtained using Balaban's mean square distance index and the accuracy of prediction was found to be about 80% in the active range [2].

The connective eccentricity index (CEI) of a graph G was defined as

$$\xi^{ce}(G) = \sum_{u \in V} \frac{d(u)}{\varepsilon(u)} = \sum_{uv \in E} \left(\frac{1}{\varepsilon(u)} + \frac{1}{\varepsilon(v)} \right). \quad (1)$$

The upper or lower bounds for the connective eccentricity index in terms of some graph invariants such as the radius, the independence number, the vertex connectivity, the minimum degree, the maximum degree etc. were recently reported in [3, 4, 5]. In this paper, we will prove that the "greedy" caterpillar minimizes $\xi^{ce}(T)$, while the "greedy" tree maximizes $\xi^{ce}(T)$ among all trees with a given degree sequence. Moreover, we will determine the lower and upper bounds for the connective eccentricity index of an n -vertex tree with a given number of branching vertices.

2 Preliminaries

In the following, we give some transformations which will be used in the next section.

Lemma 2.1. (The transformation A) *Let u be a vertex of a graph Q with at least two vertices. For integer $a \geq 1$, G_1 is the tree obtained by attaching a star S_{a+1} at its center v to u of Q , G_2 is the tree obtained by attaching $a + 1$ pendent vertices to u of Q (see Figure 1). Then $\xi^{ce}(G_2) \geq \xi^{ce}(G_1)$.*

Proof. In the graph G_1 , we have $\varepsilon_{G_1}(v) = \varepsilon_Q(u) + 1 \geq \varepsilon_{G_1}(u) = \max\{2, \varepsilon_Q(u)\}$. It is easy to see from Figure 1 that $\varepsilon_{G_1}(w) \geq \varepsilon_{G_2}(w)$ for $w \in V$ and $d_{G_1}(w) = d_{G_2}(w)$ for $w \in V - \{u, v\}$. By the definition of $\xi^{ce}(G)$, we have

$$\begin{aligned} \xi^{ce}(G_2) - \xi^{ce}(G_1) &\geq \frac{d_{G_2}(u)}{\varepsilon_{G_2}(u)} - \frac{d_{G_1}(u)}{\varepsilon_{G_1}(u)} + \frac{d_{G_2}(v)}{\varepsilon_{G_2}(v)} - \frac{d_{G_1}(v)}{\varepsilon_{G_1}(v)} \\ &\geq \frac{d_{G_2}(u)}{\varepsilon_{G_1}(u)} - \frac{d_{G_1}(u)}{\varepsilon_{G_1}(u)} + \frac{d_{G_2}(v)}{\varepsilon_{G_1}(v)} - \frac{d_{G_1}(v)}{\varepsilon_{G_1}(v)} \\ &= a \left(\frac{1}{\varepsilon_{G_1}(u)} - \frac{1}{\varepsilon_{G_1}(v)} \right) \\ &\geq 0. \end{aligned}$$

□

Lemma 2.2. (The transformation B) *Let w be a vertex of a nontrivial connected graph G . For nonnegative integers p and q , $G(p, q)$ denotes the graph obtained from G by attaching to the vertex w pendent paths $P = wv_1v_2 \cdots v_p$ and $Q = wu_1u_2 \cdots u_q$ of lengths p and q , respectively. If $p \geq q \geq 1$, then $\xi^{ce}(G(p, q)) \geq \xi^{ce}(G(p+1, q-1))$.*

Proof. (1) For $q > 1$, let $G(p+1, q-1)$ be obtained from $G(p, q)$ by deleting the edge $u_q u_{q-1}$ and adding an edge $v_p u_q$. We have $\varepsilon_{G(p,q)}(v_p) \geq \varepsilon_{G(p+1,q-1)}(v_p)$, $\varepsilon_{G(p,q)}(u_{q-1}) \leq \varepsilon_{G(p+1,q-1)}(u_{q-1})$ and $\varepsilon_{G(p,q)}(u_q) \leq \varepsilon_{G(p+1,q-1)}(u_q)$. If $t \in V - \{u_p, v_{q-1}\}$, then $d_{G(p,q)}(t) = d_{G(p+1,q-1)}(t)$ and $\varepsilon_{G(p,q)}(t) \leq \varepsilon_{G(p+1,q-1)}(t)$. So,

$$\begin{aligned} & \xi^{ce}(G(p+1, q-1)) - \xi^{ce}(G(p, q)) \\ & \leq \frac{d_{G(p+1,q-1)}(v_p)}{\varepsilon_{G(p+1,q-1)}(v_p)} - \frac{d_{G(p,q)}(v_p)}{\varepsilon_{G(p,q)}(v_p)} + \frac{d_{G(p+1,q-1)}(u_{q-1})}{\varepsilon_{G(p+1,q-1)}(u_{q-1})} - \frac{d_{G(p,q)}(u_{q-1})}{\varepsilon_{G(p,q)}(u_{q-1})} + \frac{d_{G(p+1,q-1)}(u_q)}{\varepsilon_{G(p+1,q-1)}(u_q)} - \frac{d_{G(p,q)}(u_q)}{\varepsilon_{G(p,q)}(u_q)} \\ & = \frac{2}{\varepsilon_{G(p+1,q-1)}(v_p)} - \frac{1}{\varepsilon_{G(p,q)}(v_p)} + \frac{1}{\varepsilon_{G(p+1,q-1)}(u_{q-1})} - \frac{2}{\varepsilon_{G(p,q)}(u_{q-1})} + \frac{1}{\varepsilon_{G(p+1,q-1)}(u_q)} - \frac{1}{\varepsilon_{G(p,q)}(u_q)} \\ & \leq \frac{3}{\varepsilon_{G(p+1,q-1)}(u_q)} - \frac{3}{\varepsilon_{G(p,q)}(v_p)} \\ & \leq 0. \end{aligned}$$

(2) For $q = 1$, let $G(p+1, 0)$ be obtained from $G(p, 1)$ by deleting the edge $u_1 w$ and adding an edge $v_p u_1$. If $t \in V - \{u_p, w\}$, then $d_{G(p,q)}(t) = d_{G(p+1,q-1)}(t)$ and $\varepsilon_{G(p,q)}(t) \leq \varepsilon_{G(p+1,q-1)}(t)$. So, we have

$$\begin{aligned} & \xi^{ce}(G(p+1, 0)) - \xi^{ce}(G(p, 1)) \\ & \leq \frac{d_{G(p+1,0)}(v_p)}{\varepsilon_{G(p+1,0)}(v_p)} - \frac{d_{G(p,1)}(v_p)}{\varepsilon_{G(p,1)}(v_p)} + \frac{d_{G(p+1,0)}(w)}{\varepsilon_{G(p+1,0)}(w)} - \frac{d_{G(p,1)}(w)}{\varepsilon_{G(p,1)}(w)} + \frac{d_{G(p+1,0)}(u_1)}{\varepsilon_{G(p+1,0)}(u_1)} - \frac{d_{G(p,1)}(u_1)}{\varepsilon_{G(p,1)}(u_1)} \\ & = \frac{2}{\varepsilon_{G(p+1,0)}(v_p)} - \frac{1}{\varepsilon_{G(p,1)}(v_p)} + \frac{d(w)-1}{\varepsilon_{G(p+1,0)}(w)} - \frac{d(w)}{\varepsilon_{G(p,1)}(w)} + \frac{1}{\varepsilon_{G(p+1,0)}(u_1)} - \frac{1}{\varepsilon_{G(p,1)}(u_1)} \\ & \leq \frac{3}{\varepsilon_{G(p+1,0)}(u_1)} - \frac{3}{\varepsilon_{G(p,q)}(v_p)} \\ & \leq 0. \end{aligned}$$

From above, the result is proved. \square

By using Lemma 2.1 and Lemma 2.2, we can obtain the following result directly.

Proposition 2.3. *Let T be a tree with $n \geq 6$ vertices and $T \neq S_n, P_n, T_1, T_2$ (depicted in Figure 2). Then*

$$\xi^{ce}(S_n) > \xi^{ce}(T_2) > \xi^{ce}(T) > \xi^{ce}(T_1) > \xi^{ce}(P_n).$$

3 The connective eccentricity index of trees with a given degree sequence

Given a degree sequence, let \mathcal{T} be the class of trees that realize this degree sequence. We will determine the trees which maximize or minimize the connective eccentricity index in \mathcal{T} , and will compare the maximal values of the connective eccentricity index for different degree sequences. Note that a sequence (d_1, d_2, \dots, d_n) of positive integers is a degree sequence of a tree if and only if $\sum_{i=1}^n d_i = 2(n-1)$.

In the following, we firstly show that the greedy caterpillar minimize the connective eccentricity index in \mathcal{T} .

In [11], Wang gave the definition of the *greedy caterpillar*. Greedy caterpillars are not unique with given a degree sequence.

Definition 3.1. [11] For $n \geq 3$, let $\bar{d} = (d_1, d_2, \dots, d_n)$ be the non-increasing degree sequence of a tree with $d_k > 1$ and $d_{k+1} = 1$ for some $k \in \{1, 2, \dots, n-2\}$. The greedy caterpillar, T , is constructed as follows:

- Start with a path $P = z_1 z_2 \dots z_k$.
- Let $\phi : \{z_i\}_{i=1}^k \rightarrow \{d_i\}_{i=1}^k$ be a one-to-one function such that, for each pair $i, j \in [k]$, if $\varepsilon_P(z_i) > \varepsilon_P(z_j)$, then $\phi(z_i) \geq \phi(z_j)$.
- For each $i \in \{2, 3, \dots, k-1\}$, attach $\phi(z_i) - 2$ pendant vertices to z_i . For $i \in \{1, k\}$, attach $\phi(z_i) - 1$ pendant vertices to z_i .

Theorem 3.2. Among trees with a given tree degree sequence, the greedy caterpillar has the minimum the connective eccentricity index.

Proof. Fix a degree sequence $\bar{d} = (d_1, \dots, d_n)$ which is written in the form described in Definition 3.1. Let \mathcal{T} be the collection of trees with degree sequence \bar{d} , and $T \in \mathcal{T}$ such that $\xi^{ce}(T) = \min_{F \in \mathcal{T}} \xi^{ce}(F)$. We first show that T is a caterpillar.

By contradiction, suppose T is not a caterpillar. Let $P_T(u, v) = uu_1u_2 \dots u_kv$ be a longest path in T . Let $x \in \{1, 2, \dots, k\}$ be the least integer such that u_x has a non-leaf neighbor w not on $P_T(u, v)$. Then $x \neq 1$ for the maximality of $P_T(u, v)$. Let W be the component containing w in $T - \{u_x w\}$.

Create a new tree T' from T by replacing each edge of the form zw in W with the edge zu (see Figure 3). Notice that T and T' have the same degree sequence. However, for any vertex $s \in (V(T) \setminus V(W)) \cup \{w\}$, $\varepsilon_{T'}(s) \geq \varepsilon_T(s)$ since $P_T(u, v)$ is a longest path in T . For any vertex $r \in V(W) - w$, we have

$$\varepsilon_{T'}(r) = d(r, u) + d(u, v) > d(u, v) \geq \varepsilon_T(r).$$

By the definition of the connective eccentricity index, we have $\xi^{ce}(T') < \xi^{ce}(T)$, a contradiction.

Now, we will show that T is a greedy caterpillar. By contradiction, suppose T is not a greedy caterpillar. Since T is a caterpillar with internal vertices forming path $P = u_1 u_2 \dots u_k$, the eccentricity of any internal vertex is independent of the interval vertex degree assignments. There must be $i, j \in \{1, 2, \dots, k\}$ with $d_T(u_i) < d_T(u_j)$ and $\varepsilon_T(u_i) > \varepsilon_T(u_j)$.

Create a new tree T'' from T by replacing each edge of the form $u_j w, u_i t$ (w, t be the pendant vertices of u_j, u_i , respectively) with the edge $u_i w, u_j t$. Notice that T and T'' have the same degree sequence and $d_{T''}(u_i) = d_T(u_j)$, $d_{T''}(u_j) = d_T(u_i)$, $\varepsilon_{T''}(u_i) = \varepsilon_T(u_i)$, $\varepsilon_{T''}(u_j) = \varepsilon_T(u_j)$. We have

$$\begin{aligned}
\xi^{ce}(T'') - \xi^{ce}(T) &< \frac{d_{T''}(u_i)}{\varepsilon_{T''}(u_i)} + \frac{d_{T''}(u_j)}{\varepsilon_{T''}(u_j)} - \frac{d_T(u_i)}{\varepsilon_T(u_i)} - \frac{d_T(u_j)}{\varepsilon_T(u_j)} \\
&= (d_T(u_j) - d_T(u_i)) \left(\frac{1}{\varepsilon_T(u_i)} - \frac{1}{\varepsilon_T(u_j)} \right) \\
&< 0
\end{aligned}$$

a contradiction. \square

Next, we will show that the greedy tree maximize the connective eccentricity index in \mathcal{T} .

Each tree is rooted at a vertex (while the root has no bearing on the connective eccentricity index, we use the added structure to direct our conversation). The height of a vertex is the distance to the root, and the tree's height, $h = h(T)$, is the maximum of all heights of vertices. We start with some definitions.

Definition 3.3. [12] *In a rooted tree, the list of multisets L_i of degrees of vertices at height i , starting with L_0 containing the degree of the root vertex, is called the level-degree sequence of the rooted tree.*

Let $|L_i|$ be the number of entries in L_i . It is easy to see that a list of multisets is the level degree sequence of a rooted tree if and only if (1) the multiset $\bigcup_i L_i$ is a tree degree sequence; (2) $|L_0| = 1$; and (3) $\sum_{d \in L_0} d = |L_1|$ and for all $i \geq 1$, $\sum_{d \in L_i} (d - 1) = |L_{i+1}|$.

In a rooted tree, the *down-degree* of the root is equal to its degree. The down degree of any other vertex is its degree minus one.

Definition 3.4. [12] *Given the level-degree sequence of a rooted tree, the level-greedy rooted tree for this level-degree sequence is built as follows: (1) For each $i \in \{1, 2, \dots, n\}$, place $|L_i|$ vertices in level i and to each vertex, from left to right, assign a degree from L_i in non-increasing order; (2) For $i \in \{1, 2, \dots, n - 1\}$, from left to right, join the next vertex in L_i whose down-degree is d to the first d so far unconnected vertices on level L_{i+1} . Repeat for $i + 1$.*

Definition 3.5. [12] *Given a tree degree sequence (d_1, d_2, \dots, d_n) in non-increasing order, the greedy tree for this degree sequence is the level-greedy tree for the level-degree sequence that has $L_0 = \{d_1\}$, $L_1 = \{d_2, \dots, d_{d_1+1}\}$ and for each $i > 1$,*

$$|L_i| = \sum_{d \in L_{i-1}} (d - 1)$$

with every entry in L_i at most as large as every entry in L_{i-1} .

A greedy tree with the degree sequence $(4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 2, 2, 1, \dots, 1)$ is shown in Figure 4.

Lemma 3.6. *Among all the trees with a given level-degree sequence, the level-greedy tree maximizes the connective eccentricity index.*

Proof. By induction on the number of vertices, the base case with one vertex is trivial.

Let T be a rooted tree with the given level-degree sequence and maximize the connective eccentricity index (i.e. T is optimal). For vertices $w \in T_1$ and $u \in T - T_1$, both of height j (See Figure 5), we notice that $\varepsilon_T(u) = j + h$, $\varepsilon_T(w) = \max\{j + h', \varepsilon_{T_1}(w)\} \leq \varepsilon_T(u)$. Suppose for contradiction that $d_T(u) > d_T(w)$. Create a new tree T' by moving $d_T(u) - d_T(w)$ children of u and their descendants to adoptive parent w . This effectively switches the degrees of u and w while maintaining the level degree sequence.

While $\varepsilon_{T'}(u) = \varepsilon_T(u)$, notice that h' does not increase and $\varepsilon_T(x) \geq \varepsilon_{T'}(x)$ for all $x \in V$. Since $\varepsilon_{T'}(w) \leq \max\{j + h', \varepsilon_{T_1}(w)\} = \varepsilon_T(w)$, we have

$$\begin{aligned} \xi^{ce}(T') - \xi^{ce}(T) &\geq \frac{d_{T'}(u)}{\varepsilon_{T'}(u)} + \frac{d_{T'}(w)}{\varepsilon_{T'}(w)} - \frac{d_T(u)}{\varepsilon_T(u)} - \frac{d_T(w)}{\varepsilon_T(w)} \\ &\geq (d_T(w) - d_T(u))\left(\frac{1}{\varepsilon_T(u)} - \frac{1}{\varepsilon_T(w)}\right) \\ &> 0 \end{aligned}$$

a contradiction to the optimality of T . Otherwise, T' and T are both optimal trees. In this case, we can repeat this shifting of degrees for pairs of vertices of height 1, followed by pairs of vertices of height 2, and so on until we either meet a contradiction or construct an optimal tree in which $d(u) \leq d(w)$ for all $w \in T_1$ and $u \in T - T_1$ of the same height.

Now, we have a partition of the level-degree sequence for T into the level-degree sequences for $T - T_1$. By the inductive hypothesis, we may assume that both T_1 and $T - T_1$ are level-greedy trees on their level-degree sequences. As a result, T is a level-greedy tree. \square

The next theorem also yields a stronger result than merely the connective eccentricity index among trees with a given degree sequence.

Theorem 3.7. *Among all trees with a given degree sequence, the greedy tree has the maximal connective eccentricity index.*

Proof. Let $\bar{d} = (d_1, d_2, \dots, d_n)$ be given degree sequence in non-increasing order and T^* the tree with the maximal connective eccentricity index with the given degree sequence.

Take a longest path in T^* and a center vertex v of this path as the root of T^* . In $T^* - \{v\}$, let T_1 be the component containing the leaf with the greatest height. By

our choice of the root, if h is the height of T_1 , then $T - T_1$ has height $h' \in \{h-1, h\}$. The vertex set V of T^* can be divided into h subsets $V = V_0 \cup V_1 \cdots \cup V_h$, where $V_0 = \{v\}$, $V_1 = \{u_1, \dots, u_{d(v)}\}$ and for each $i > 1$,

$$|V_i| = \sum_{u \in V_{i-1}} d(u) - |V_{i-1}|.$$

By Lemma 3.6, T^* is a level greedy tree. Next, we will prove that degree of every entry in V_i at most as large as degree of every entry in V_{i-1} .

Suppose that there are $V_i = \{w_1, \dots, w_k\}$ and $V_{i-1} = \{v_1, \dots, v_t\}$ such that $d_{T^*}(w_1) > d_{T^*}(v_t)$ and $w_1 \in T_1$. Create a new tree T' by moving $d_{T^*}(w_1) - d_{T^*}(v_t)$ children of w_1 and their descendants to adoptive parent v_t with the height of T_1 no change. This effectively switches the degrees of w_1 and v_t while maintaining the degree sequence. We now examine two cases: $v_t \in T_1$ and $v_t \in T^* - T_1$.

Case I. $v_t \in T_1$. Note that $\varepsilon_{T^*}(w_1) = \varepsilon_{T'}(w_1) \geq \varepsilon_{T^*}(v_t) = \varepsilon_{T'}(v_t)$ and $\varepsilon_{T'}(x) \leq \varepsilon_{T^*}(x)$ for all $x \in V$, we have

$$\begin{aligned} \xi^{ce}(T') - \xi^{ce}(T^*) &\geq \frac{d_{T'}(v_t)}{\varepsilon_{T'}(v_t)} + \frac{d_{T'}(w_1)}{\varepsilon_{T'}(w_1)} - \frac{d_{T^*}(v_t)}{\varepsilon_{T^*}(v_t)} - \frac{d_{T^*}(w_1)}{\varepsilon_{T^*}(w_1)} \\ &\geq (d_{T^*}(w_1) - d_{T^*}(v_t)) \left(\frac{1}{\varepsilon_{T^*}(v_t)} - \frac{1}{\varepsilon_{T^*}(w_1)} \right) \\ &> 0 \end{aligned}$$

a contradiction to the optimality of T .

Case II. $v_t \in T^* - T_1$. If $h' = h$, we notice that $\varepsilon_{T'}(w_1) = \varepsilon_{T^*}(w_1) = i + h > \varepsilon_{T'}(v_t) = \varepsilon_{T^*}(v_t) = i - 1 + h$ and $\varepsilon_{T'}(x) \leq \varepsilon_{T^*}(x)$ for all $x \in V$, then

$$\begin{aligned} \xi^{ce}(T') - \xi^{ce}(T^*) &\geq \frac{d_{T'}(v_t)}{\varepsilon_{T'}(v_t)} + \frac{d_{T'}(w_1)}{\varepsilon_{T'}(w_1)} - \frac{d_{T^*}(v_t)}{\varepsilon_{T^*}(v_t)} - \frac{d_{T^*}(w_1)}{\varepsilon_{T^*}(w_1)} \\ &\geq (d_{T^*}(w_1) - d_{T^*}(v_t)) \left(\frac{1}{\varepsilon_{T^*}(v_t)} - \frac{1}{\varepsilon_{T^*}(w_1)} \right) \\ &> 0 \end{aligned}$$

a contradiction to optimality of T .

If $h' = h - 1$, we notice that $\varepsilon_{T'}(w_1) = \varepsilon_{T^*}(w_1) = \varepsilon_{T'}(v_t) = \varepsilon_{T^*}(v_t) = i - 1 + h$ and $\varepsilon_{T'}(x) = \varepsilon_{T^*}(x)$ for all $x \in V$, then $\xi^{ce}(T') = \xi^{ce}(T^*)$.

In conclusion, we have that the greedy tree has the maximal connective eccentricity index among the trees with a given degree sequence. \square

Remark 3.8. *Such extremal trees are not necessarily unique. In fact, the greedy tree give a more stronger restriction than what we needed, as stated in the theorem, while still not being the unique structure.*

In the following, we will compare the connective eccentricity indices of greedy trees with different degree sequences.

Definition 3.9. Let $\pi' = (d'_1, \dots, d'_n)$ and $\pi'' = (d''_1, \dots, d''_n)$ be two non-increasing tree degree sequences. π'' is said to majorize π' , denoted $\pi' \triangleleft \pi''$, if for $k \in \{1, 2, \dots, n-1\}$

$$\sum_{i=0}^k d'_i \leq \sum_{i=0}^k d''_i \text{ and } \sum_{i=0}^n d'_i = \sum_{i=0}^n d''_i.$$

Lemma 3.10. [13] Let $\pi' = (d'_1, \dots, d'_n)$ and $\pi'' = (d''_1, \dots, d''_n)$ be two non-increasing tree degree sequences. If $\pi' \triangleleft \pi''$, then there exists a series of (non-increasing) tree degree sequences $\pi^{(i)} = (d_1^{(i)}, \dots, d_n^{(i)})$ for $1 \leq i \leq m$ such that

$$\pi' = \pi^{(1)} \triangleleft \pi^{(2)} \triangleleft \dots \triangleleft \pi^{(m-1)} \triangleleft \pi^{(m)} = \pi''.$$

In addition, each $\pi^{(i)}$ and $\pi^{(i+1)}$ differ at exactly two entries, say the j and k entries, $j < k$, where $d_j^{(i+1)} = d_j^{(i)} + 1$ and $d_k^{(i+1)} = d_k^{(i)} - 1$.

Theorem 3.11. Let $\pi' = (d'_1, \dots, d'_n)$ and $\pi'' = (d''_1, \dots, d''_n)$ be two non-increasing greedy tree degree sequences. If $\pi' \triangleleft \pi''$, then

$$\xi^{ce}(T_{\pi'}^*) \leq \xi^{ce}(T_{\pi''}^*)$$

where T_{π}^* is the greedy tree for degree sequence π .

Proof. According to Lemma 3.10, it suffices to compare the connective eccentricity indices of two greedy trees whose degree sequences differ in two entries, each by exactly 1, i.e., we can assume that

$$\pi' = (d'_1, \dots, d'_n) \triangleleft (d''_1, \dots, d''_n) = \pi''$$

with $d''_j = d'_j + 1$, $d''_k = d'_k - 1$ for some $j < k$ and all other entries are the same.

Let u and v be the vertices corresponding to d'_j and d'_k , respectively, and w be a child of v in $T_{\pi'}^*$ (see Figure 6). Construct $T_{\pi''}$ from $T_{\pi'}^*$ by removing the edge vw and adding edge uw . Note that $T_{\pi''}$ has the degree sequence π'' , and by Theorem 3.7

$$\xi^{ce}(T_{\pi''}^*) \geq \xi^{ce}(T_{\pi''}).$$

On the other hand, from the definition of the connective eccentricity index, we have

$$\begin{aligned} \xi^{ce}(T_{\pi''}) - \xi^{ce}(T_{\pi'}^*) &\geq \frac{d''_v}{\varepsilon_{T_{\pi''}}(v)} - \frac{d'_v}{\varepsilon_{T_{\pi'}^*}(v)} + \frac{d''_w}{\varepsilon_{T_{\pi''}}(w)} - \frac{d'_w}{\varepsilon_{T_{\pi'}^*}(w)} + \frac{d''_u}{\varepsilon_{T_{\pi''}}(u)} - \frac{d'_u}{\varepsilon_{T_{\pi'}^*}(u)} \\ &\geq \frac{d'_v-1}{\varepsilon_{T_{\pi'}^*}(v)} - \frac{d'_v}{\varepsilon_{T_{\pi'}^*}(v)} + \frac{1}{\varepsilon_{T_{\pi'}^*}(w)} - \frac{1}{\varepsilon_{T_{\pi'}^*}(w)} + \frac{d'_u+1}{\varepsilon_{T_{\pi'}^*}(u)} - \frac{d'_u}{\varepsilon_{T_{\pi'}^*}(u)} \\ &= \frac{1}{\varepsilon_{T_{\pi'}^*}(u)} - \frac{1}{\varepsilon_{T_{\pi'}^*}(v)}. \end{aligned}$$

By the proof of Theorem 3.7, we can see $\varepsilon_{T_{\pi'}^*}(u) \leq \varepsilon_{T_{\pi'}^*}(v)$. So, $\xi^{ce}(T_{\pi''}) \geq \xi^{ce}(T_{\pi'}^*)$.

Hence, $\xi^{ce}(T_{\pi''}^*) \geq \xi^{ce}(T_{\pi''}) \geq \xi^{ce}(T_{\pi'}^*)$. \square

4 The connective eccentricity index of trees with a given number of branching vertices

A vertex of a tree T with degree 3 or greater is called a branching vertex of T . For such a tree T , it is easy to find that $r \leq \frac{n}{2} - 1$. Note that each tree different from the path possesses at least one branching vertices. In the following, we will find a lower bound and an upper bound for the connective eccentricity index of an n -vertex tree with a given number of branching vertices.

Let $\mathcal{BT}_{n,r}$ be the set of all n -vertex trees with exactly r branching vertices. $F(n, r)$ is the greedy caterpillar with degree sequence $\bar{d} = (\overbrace{3, \dots, 3}^r, 2, \dots, 2, 1, \dots, 1)$, and $B(n, r)$ is the greedy tree with degree sequence $\bar{d} = (\overbrace{n - 2r + 1, 3, \dots, 3}^r, 1, \dots, 1)$, see Figure 7. Clearly, $F(n, r), B(n, r) \in \mathcal{BT}_{n,r}$ and $B(n, 1) = S_n$.

Theorem 4.1. *If $T \in \mathcal{BT}_{n,r}$ and $1 \leq r \leq \frac{n}{2} - 1$, then*

$$\xi^{ce}(T) \geq \xi^{ce}(F(n, r))$$

with equality if and only if $T = F(n, r)$.

Proof. Let $T \in \mathcal{BT}_{n,r}$ be a tree with the maximal connective eccentricity index. $P = v_0 v_1 \dots v_t$ is a longest path in T , and u_1, u_2, \dots, u_r are all branching vertices of T .

First, we show that $d(v) \leq 3$ for $u \in V(T)$. If there is a vertex u_i with $d(u_i) > 3$ and w is its neighbor and $w \notin P$ (See Figure 8). Create a new tree T' (See Figure 8) from T by replacing the edge $u_i w$ with $v_t w$. Notice that T and T' have the same number of branch vertices, and $\varepsilon_{T'}(s) \geq \varepsilon_T(s)$ for any vertex $s \in V$ since P is a longest path in T . For any vertex $s \in V - u_i$, $d_{T'}(s) = d_T(s)$ and $d_{T'}(u_i) = d_T(u_i) - 1$. So, we have

$$\begin{aligned} \xi^{ce}(T) - \xi^{ce}(T') &\geq \frac{d_T(u_i)}{\varepsilon_T(u_i)} - \frac{d_{T'}(u_i)}{\varepsilon_{T'}(u_i)} + \frac{d_T(w)}{\varepsilon_T(w)} - \frac{d_{T'}(w)}{\varepsilon_{T'}(w)} \\ &\geq \frac{d_T(u_i)}{\varepsilon_{T'}(u_i)} - \frac{d_{T'}(u_i)}{\varepsilon_{T'}(u_i)} + \frac{d_T(w)}{\varepsilon_{T'}(w)} - \frac{d_{T'}(w)}{\varepsilon_{T'}(w)} \\ &= \frac{1}{\varepsilon_{T'}(u_i)} > 0 \end{aligned}$$

a contradiction to the extremal property of T .

From above, we know that T is a tree with the degree sequence $\bar{d} = (\overbrace{3, \dots, 3}^r, 2, \dots, 2, 1, \dots, 1)$. By Theorem 3.2, we have the greedy caterpillar with the degree sequence $\bar{d} = (\overbrace{3, \dots, 3}^r, 2, \dots, 2, 1, \dots, 1)$. The result is true. \square

Theorem 4.2. *If $T \in \mathcal{BT}_{n,r}$ and $1 \leq r \leq \frac{n}{2} - 1$, then*

$$\xi^{ce}(T) \leq \xi^{ce}(B(n, r)).$$

Proof. Let $T \in \mathcal{BT}_{n,r}$ be a tree with the minimal connective eccentricity index. Note that every pendant path in T is a pendant edge by Lemma 2.2.

We first show that T has no vertex of degree two. If v is a vertex of degree two in T , then there is a branching vertex u in T such that $\varepsilon(u) > \varepsilon(v)$ and its neighbors except one are pendant vertices v_1, \dots, v_k , where $k = \deg(u) - 1$ (see Figure 9). Create a new tree T' from T by replacing edges uv_i ($1 \leq i \leq k$) with vu_i ($1 \leq i \leq k$). Notice that $T' \in \mathcal{BT}_{n,r}$ with $\varepsilon_{T'}(s) \leq \varepsilon_T(s)$ for any vertex $s \in V(T)$, $d_{T'}(s) = d_T(s)$ for any vertex $s \in V - u, v$ and $d_{T'}(u) = d_T(u) - k = 1$, $d_{T'}(v) = d_T(v) + k = 2 + k$, $\varepsilon_{T'}(u) \leq \varepsilon_{T'}(v)$. So, we have

$$\begin{aligned} \xi^{ce}(T) - \xi^{ce}(T') &\leq \frac{d_T(u)}{\varepsilon_T(u)} - \frac{d_{T'}(u)}{\varepsilon_{T'}(u)} + \frac{d_T(v)}{\varepsilon_T(v)} - \frac{d_{T'}(v)}{\varepsilon_{T'}(v)} \\ &\leq \frac{d_T(u)}{\varepsilon_{T'}(u)} - \frac{d_{T'}(u)}{\varepsilon_{T'}(u)} + \frac{d_T(v)}{\varepsilon_{T'}(v)} - \frac{d_{T'}(v)}{\varepsilon_{T'}(v)} \\ &= k\left(\frac{1}{\varepsilon_{T'}(u)} - \frac{1}{\varepsilon_{T'}(v)}\right) < 0 \end{aligned}$$

a contradiction to the extremal property of T .

From above, we know that T is a tree with degree sequence $\bar{d} = (d_1, \dots, d_r, 1, \dots, 1)$. By Theorem 3.11 and Theorem 3.7, we have the greedy tree with the degree sequence $\bar{d} = (\overbrace{n-2r+1, 3, \dots, 3}^r, 1, \dots, 1)$. The result holds. \square

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